Week 12

Properties of definite integrals

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

s,t are constants

4. $\int_a^b (s f(x) + t g(x))dx = s\int_a^b f(x)dx + t\int_a^b g(x)dx$

5. If $f(x) \leq g(x)$ on $[a,b]$ with $a \leq b$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)dx$$

Special case: if $m \leq f(x) \leq M$, on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Picture

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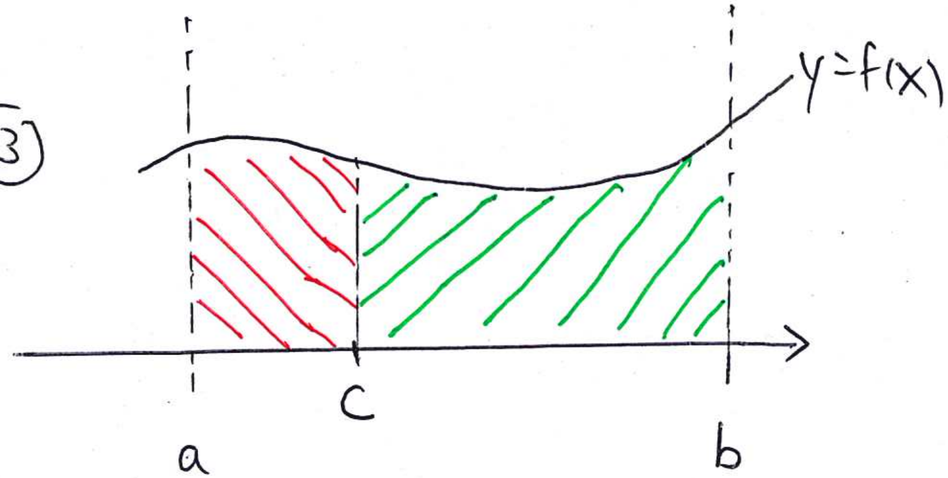
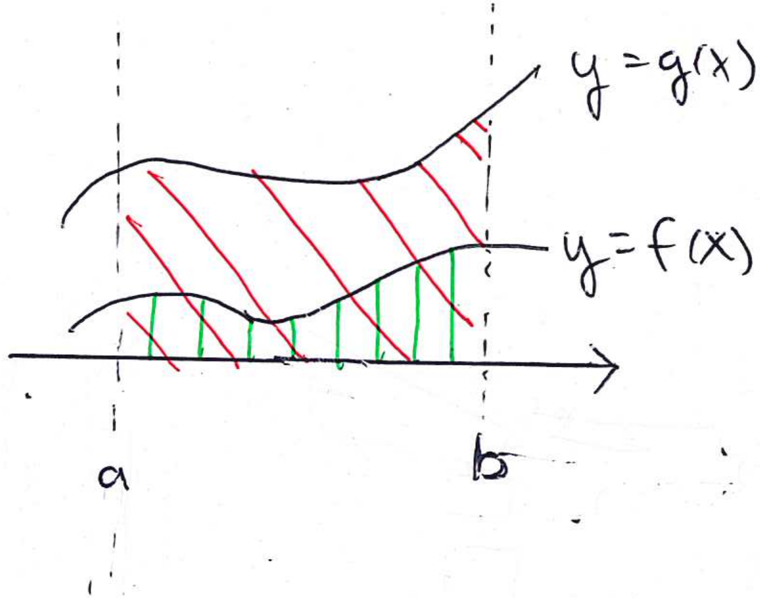


if $c \in (a,b)$

5

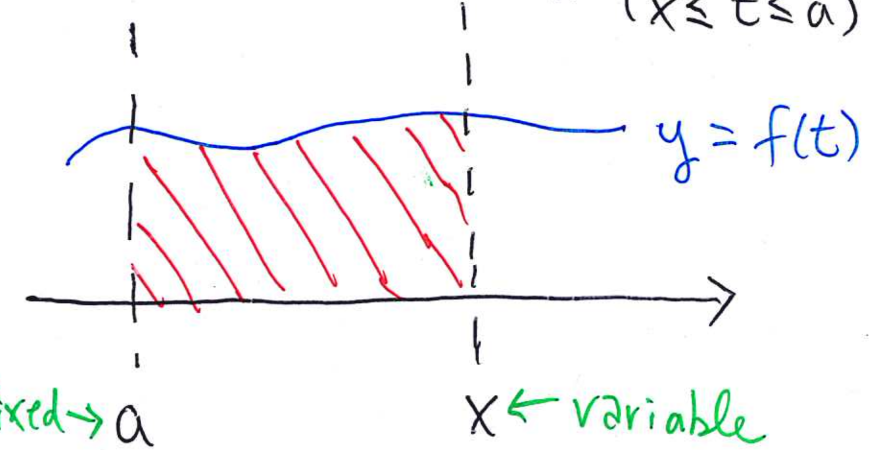


Fundamental theorem of calculus

Area as a function

$$\int_a^x f(t)\,dt = \text{Area under graph of } f(t) \text{ on } a \le t \le x$$

or $(x \le t \le a)$



1st FTC: Let $f(t)$ be continuous.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

same

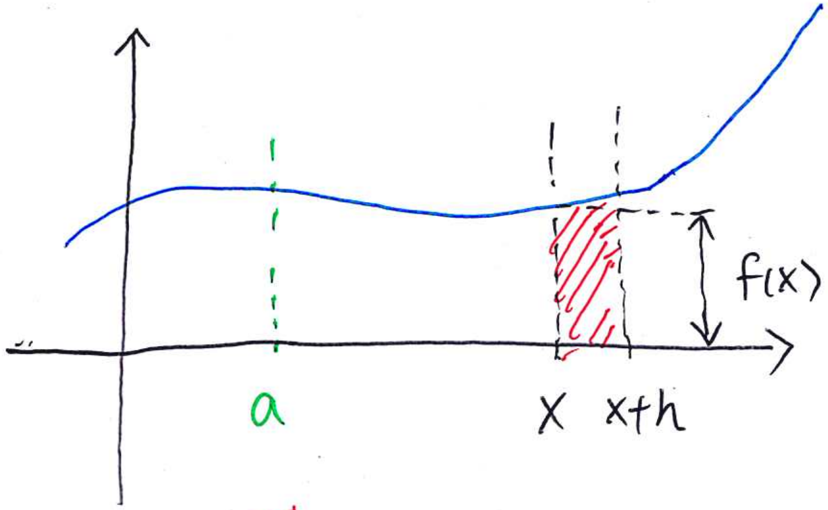
same

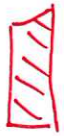
same

Idea

$$\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

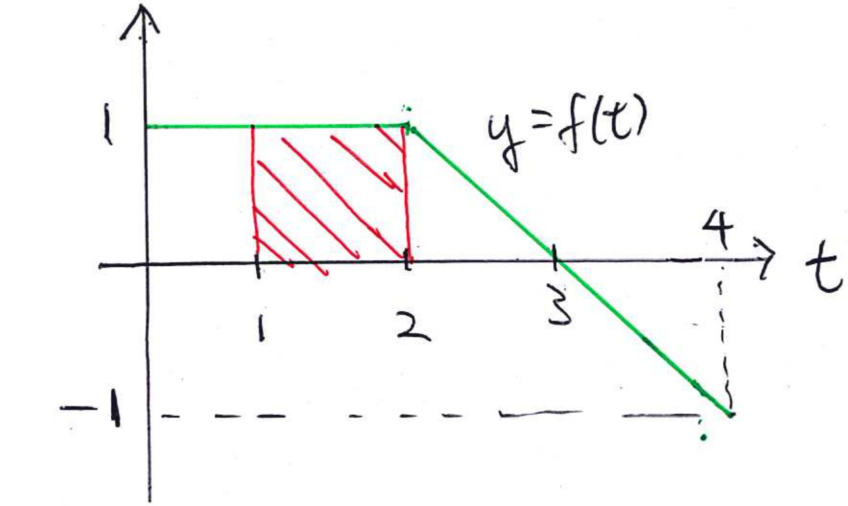
$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$$



Area of  $\approx h\,f(x)$

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt \approx f(x)$$

eg



Let $F(x) = \int_1^x f(t)\,dt$

$F(1) = 0$

$F(2) = \int_1^2 f(t)\,dt$

$=$ Area of 

$= 1$

$F(3) = \frac{3}{2}$

$F(4) = 1 + \frac{1}{2} + (-\frac{1}{2}) = 1$

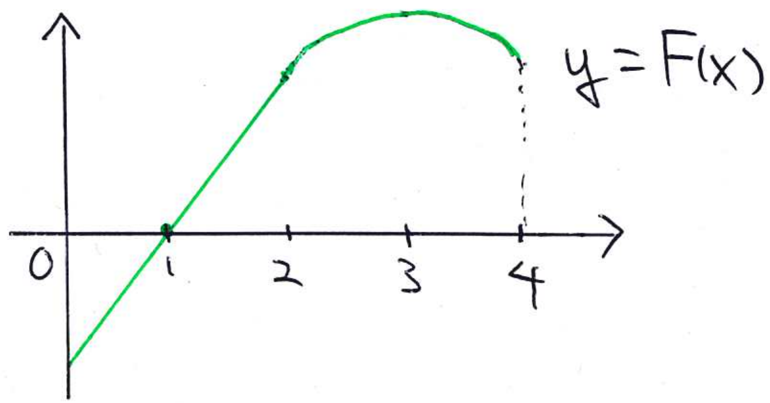
$F(0) = \int_1^0 f(t) = -\int_0^1 f(t)\,dt = -1$

FTC $\Rightarrow$ $F'(x) = f(x)$

$$\Rightarrow F'(x) = \begin{cases} > 0 & \text{if } 0 \leq x < 3 \\ = 0 & \text{if } x = 3 \\ < 0 & \text{if } 3 < x \leq 4 \end{cases}$$

$\therefore$ $F(x)$ has local maximum at $x = 3$

Graph of $F(x)$



eg $g(x)=\int_1^x \ln(t^2+3)\,dt$. $g'(3)=?$

Sol FTC $\Rightarrow g'(x)=\ln(x^2+3)$

$\therefore g'(3)=\ln(3^2+3)=\ln 12$

eg Find $g'(x)$ if

a. $g(x)=\int_{-1}^{x^2} e^t\,dt$ b. $g(x)=\int_{x^2}^{x^4}\cos(t^2)\,dt$

c. $g(x)=\int_1^x e^{2x+t^2}\,dt$

Sol a. $g'(x)=\left(\frac{d}{dx^2}\int_{-1}^{x^2} e^t\,dt\right)\frac{dx^2}{dx}$

Chain rule

$=e^{x^2}(2x)$

$=2xe^{x^2}$

b. $g(x)=\int_0^{x^4}\cos(t^2)\,dt+\int_{x^2}^0\cos(t^2)\,dt$

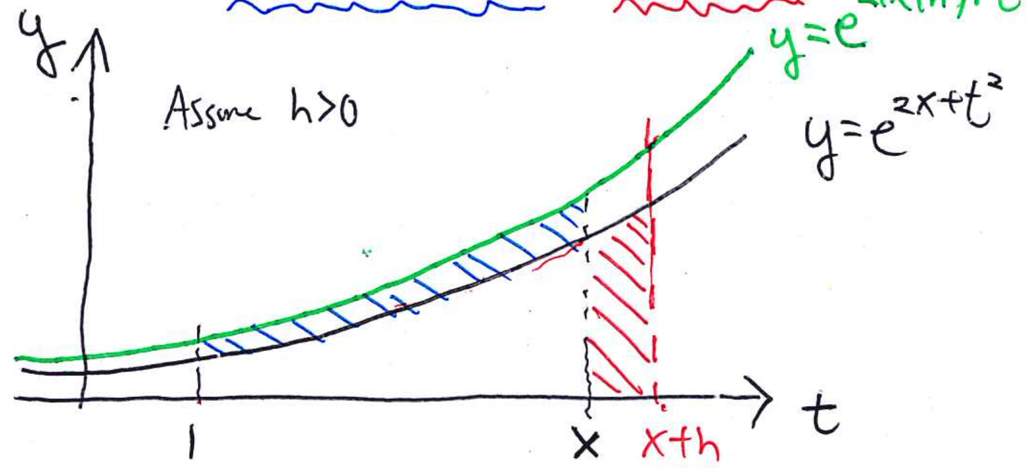
$=\int_0^{x^4}\cos(t^2)\,dt-\int_0^{x^2}\cos(t^2)\,dt$

$g'(x)=\cos((x^4)^2)(4x^3)-\cos((x^2)^2)(2x)$

$=4x^3\cos(x^8)-2x\cos(x^4)$

c. $g(x)=\int_1^x e^{2x}\cdot e^{t^2}\,dt=e^{2x}\int_1^x e^{t^2}\,dt$

$\Rightarrow g'(x)=\underbrace{2e^{2x}\int_1^x e^{t^2}\,dt}+\underbrace{e^{2x}e^{x^2}}$



2nd FTC

Suppose $F'(x) = f(x)$ on $[a,b]$

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

Rmk Notation: $\left[F(x)\right]_a^b = F(x)\Big|_a^b$

$= F(b) - F(a)$

Pf (from 1st FTC)

Let $g(x) = \int_a^x f(t)\,dt + F(a)$

1st FTC $\Rightarrow g'(x) = f(x) = F'(x)$

$\Rightarrow g(x) = F(x) + c$ for some constant

Put $x = a$, $g(a) = F(a) + c$

$F(a) = F(a) + c \Rightarrow c = 0$

$\Rightarrow F(x) = g(x)$

$\Rightarrow F(b) = g(b)$

$= \int_a^b f(t)\,dt + F(a)$

$\Rightarrow F(b) - F(a) = \int_a^b f(t)\,dt$

$= \int_a^b f(x)\,dx$

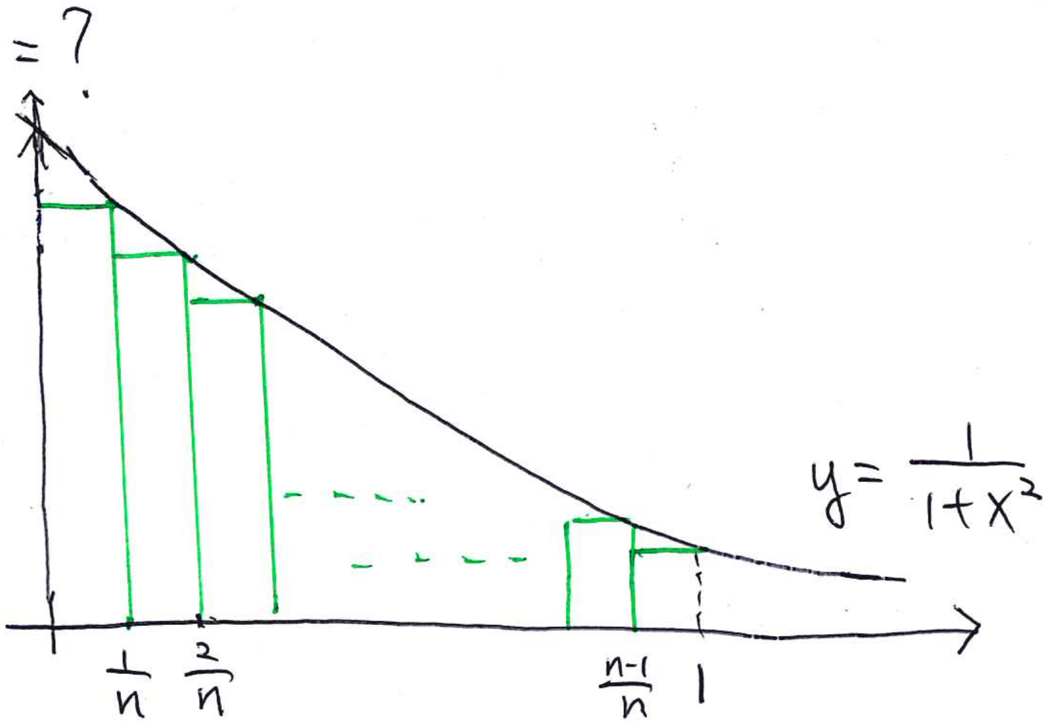
eg $\int_1^2 x^2\,dx$

$= \left[\frac{1}{3}x^3\right]_1^2$

$= \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3$

$= \frac{7}{3}$ (same as computation using Riemann Sum before) eg (Infinite sum $\longleftrightarrow$ Definite integral)

$$\lim_{n\to\infty} \frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\cdots+\frac{n}{n^2+n^2} = ?$$



Sol General term :

$$\frac{n}{n^2+k^2} = \frac{1}{n}\cdot\frac{n^2}{n^2+k^2} = \frac{1}{n}\,\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

Area of k-th rectangle $= \frac{1}{n}\,\frac{1}{1+\left(\frac{k}{n}\right)^2}$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \Big[\arctan x\Big]_0^1$$

$$= \arctan 1 - \arctan 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Ex By considering $f(x)=\frac{1}{1+x}$, show

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \ln 2$$

Definite Integral by substitution

Let $u = u(x)$

$$\int_a^b \underbrace{f(u(x))\,u'(x)\,dx}_{\text{expressed in terms of } x} = \int_{u(a)}^{u(b)} \underbrace{f(u)\,du}_{\text{in terms of } u}$$

eg $\int_0^2 x e^{x^2} dx$

Sol Let $u = x^2$ $\quad du = 2x\,dx$

When $x = 0$, $\quad u = 0^2 = 0$

When $x = 2$, $\quad u = 2^2 = 4$

$$\int_0^2 x e^{x^2} dx = \frac{1}{2}\int_0^4 e^u du$$

$$= \left[\frac{1}{2} e^u\right]_0^4 = \frac{1}{2}(e^4 - 1)$$

Alternatively

$$\int_0^2 x e^{x^2} dx = \frac{1}{2}\int_0^2 e^{x^2} dx^2$$

$$= \frac{1}{2}\left[e^{x^2}\right]_0^2$$

$$= \frac{1}{2}\left(e^{2^2} - e^{0^2}\right)$$

$$= \frac{1}{2} e^4 - 1$$

eg $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ (t-formula)

Sol Let $t = \tan\frac{x}{2}$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

When $x=0$, $t = \tan 0 = 0$

When $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x} = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{1+t^2+2t}$$

$$= \int_0^1 \frac{2dt}{(1+t)^2}$$

$$= \int_0^1 \frac{2\,d(1+t)}{(1+t)^2}$$

$$= \left[\frac{-2}{1+t}\right]_0^1$$

$$= \frac{-2}{1+1} - \frac{-2}{1+0}$$

$$= 1$$

Integration by parts for definite integral

$$\int_a^b u\,dv = \left[u\,v\right]_a^b - \int_a^b v\,du$$

eg $\int_1^n \ln x\,dx$

$$= \left[(\ln x)x\right]_1^n - \int_1^n x\,d\ln x$$

$$= n\ln n - \int_1^n x\frac{1}{x}\,dx$$

$$= n\ln n - \int_1^n 1\,dx$$

$$= n\ln n - [x]_1^n$$

$$= n\ln n - n + 1$$

Reduction formula:

e.g. Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

① Find Reduction formula for I_n

② Find I_7

Sol.

Recall: $\tan^2 x = \sec^2 x - 1$

$d\tan x = \sec^2 x \, dx$

For $n \geq 2$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan^{n-2} x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, d\tan x - I_{n-2}$$

$$= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

2. $I_7 = \frac{1}{6} - I_5$

$$= \frac{1}{6} - \frac{1}{4} + I_3$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - I_1$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \Big[\ln|\sec x| \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \ln\sqrt{2}$$

$\int \tan x \, dx$

$= \int \frac{\sin x \, dx}{\cos x}$

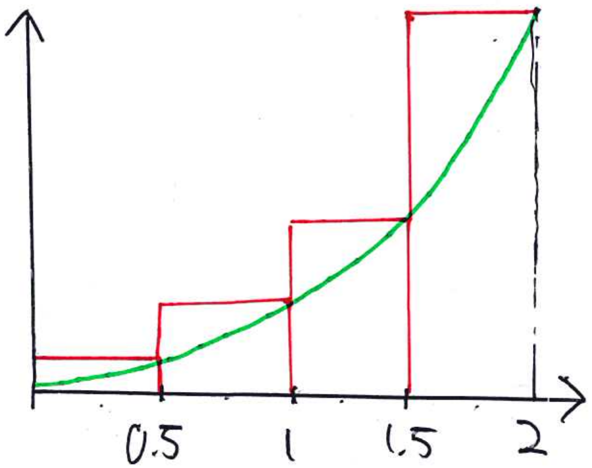
$= -\int \frac{d\cos x}{\cos x} = -\ln|\cos x| + C$

$= \ln|\sec x| + C$

Average value of a function

eg Let $f(x)=x^2$ on $[0,2]$

What is its average value of f



Estimate 1:

$n=4$

$$\text{Average} \approx \frac{1}{4}\left(f(0.5)+f(1)+f(1.5)+f(2)\right)$$

$$= \frac{1}{2}\Big[\underbrace{f(0.5)\cdot 0.5+f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5}_{\text{Total area of rectangles}}\Big]$$

length of $[0,2]$

Estimate 2: (Better)

$$\text{Average} \approx \frac{1}{8}\left[f(0.25)+f(0.5)+\cdots+f(2)\right]$$

$n=8$

$$= \frac{1}{2}\Big[\underbrace{f(0.25)\cdot 0.25+\cdots+f(2)\cdot 0.25}_{\text{Riemann Sum}}\Big]$$

Take $n \to \infty$

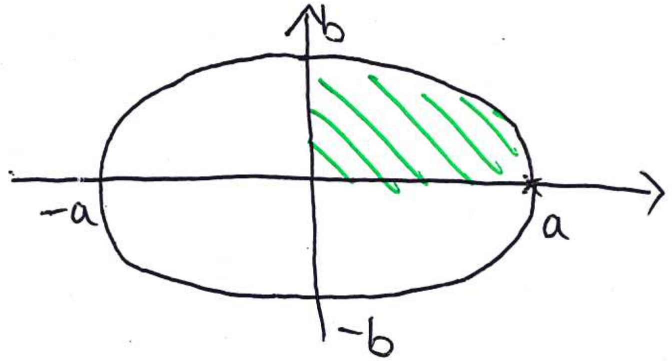
$$\text{Average value of } f = \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$$

For a general $f(x)$

$$\boxed{\text{Average value of } f \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

eg Find the area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$a, b > 0$

Sol

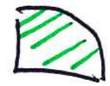
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$\Rightarrow$ Upper half $\quad y = b\sqrt{1-\frac{x^2}{a^2}}$

Lower half $\quad y = -b\sqrt{1-\frac{x^2}{a^2}}$

$$\text{Area} = \int_{-a}^{a}\left[b\sqrt{1-\frac{x^2}{a^2}}-\left(-b\sqrt{1-\frac{x^2}{a^2}}\right)\right]dx$$

Easier way for calculation

Total area = 4 Area of 

$$= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

Let $x = a\sin\theta \qquad x=a \Rightarrow \theta=\frac{\pi}{2}$

$dx = a\cos\theta\, d\theta \qquad x=0 \Rightarrow \theta = 0$

$$= 4\int_0^{\frac{\pi}{2}} b\sqrt{1-\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 2ab\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$= 2ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \pi ab$$